Fractionally Log-Concave & Sector-Stable Polynomials: Counting Planar Matchings and More

Yeganeh Alimohammadi Nima Anari Kirankumar Shiragur **Thuy-Duong "June" Vuong**

Stanford

Northwestern Theory Seminar November 2, 2022





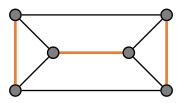


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ractionally Log-Concave & Sector Stable

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Counting Matching

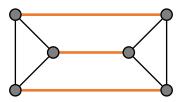


Orange edges form a matching

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Counting Matching



Orange edges form a matching Total # matchings: 2

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Q: Efficient algorithm for counting fixed-size matching in graph? A:

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- $Q{:}\ Efficient algorithm for counting fixed-size matching in graph?$
- A: Intractable
- Q: Why do we care?
- A: Counting matching might shed light on P=NP question.

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Q: Efficient algorithm for approximate counting fixed-size matching in graph? A:

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Q: Efficient algorithm for approximate counting fixed-size matching in graph? A: Open for decades

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Q: Efficient algorithm for approximate counting fixed-size matching in graph?

- A: Open for decades
- Q: Efficient algorithm for approximate counting fixed-size matching in planar graph?
- A: This work

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Background Technique

Other Applications

Counting Problems Matchings

Overview

- 1 Background
 - Counting Problems
 - Matchings
- 2 Technique
 - Reduce Counting to Sampling
 - Sampling via Random Walks
 - Fast Mixing From Sector-Stability
- **3** Other Applications

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Counting Problems Matchings

Decision vs. Counting

Given description for $L \subseteq \{0, 1\}^n$

Decision

Counting

Approximate Counting

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Decide if L is empty

 SAT: Decide if φ has a satisfying assignment

Counting Problems Matchings

Decision vs. Counting

Given description for $L \subseteq \{0,1\}^n$ e.g. *L* is set of *x* satisfying $(x_1 \lor x_2 \lor \bar{x_3}) \land (x_1 \lor x_4 \lor x_5).$

Decision

Decide if L is empty

 SAT: Decide if has a satisfying assignment

Counting

Compute |L|

 #SAT: Count # satisfying assignments

Approximate Counting

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Counting Problems Matchings

Decision vs. Counting

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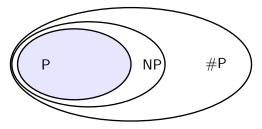
Approximate

Counting Compute \hat{Z} s.t. $0.9\hat{Z} \le |L| \le \hat{Z}$

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Counting Problems Matchings

Decision vs. Counting



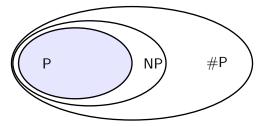
Counting is harder than Decision

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Counting Problems Matchings

Decision vs. Counting

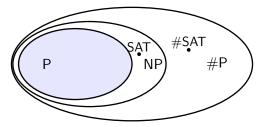


- Counting is harder than Decision
- #P-complete problems: at least as hard as all problems in #P.

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Counting Problems Matchings

Decision vs. Counting

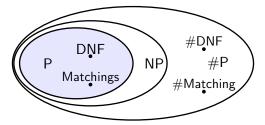


- Counting is harder than Decision
- #P-complete problems: at least as hard as all problems in #P. E.g.: #SAT

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Counting Problems Matchings

Decision vs. Counting

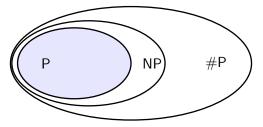


- Counting is harder than Decision
- #P-complete problems: at least as hard as all problems in #P.
- Easy to decide-hard to count: DNF formulas, matchings

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Counting Problems Matchings

Decision vs. Counting



- Counting is harder than Decision
- #P-complete problems: at least as hard as all problems in #P.
- Easy to decide-hard to count: DNF formulas, matchings
- Easier to approximate count?

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Counting Problem Matchings

Concrete example: matchings in graph

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Counting Problems Matchings

Complexity of Counting Matching

	General	Bipartite	Planar
Perfect matching			
k-matching			

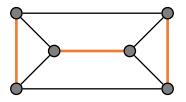
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Counting Problem Matchings

Perfect Matching

A set of n/2 edges that meets every vertex at most once.



Decision \equiv decide if G has a perfect matching (PM): efficient [Edmonds'65]

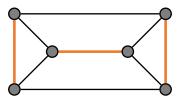
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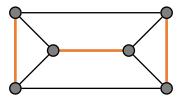
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Decision \equiv decide if *G* has a perfect matching (PM): efficient [Edmonds'65] Counting \equiv compute #PM: intractable (#P-complete [Valiant'87]) Approximate counting \equiv output \hat{Z} s.t. $0.9\hat{Z} \leq \#$ PM $\leq \hat{Z}$: open

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	General	Bipartite	Planar
Perfect matching	$\stackrel{\scriptstyle{\scriptsize (2)}}{}$		
	[Val87]		
	?		
k-matching			

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Partial results for specific graph families. Next: bipartite and planar graphs

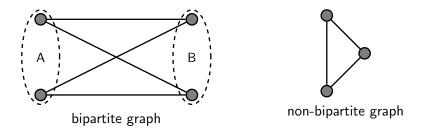
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Counting Problem Matchings

Bipartite Graphs

Bipartite graph $G = G(A \cup B, E)$: $A \cap B = \emptyset, E \subseteq A \times B$



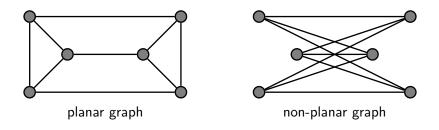
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Counting Problem Matchings

Planar Graphs

Planar graphs: can be drawn on plane without crossing edges.

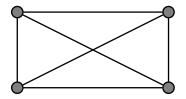


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Counting Problems Matchings

Planar Graphs

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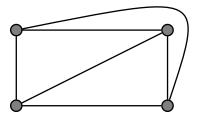


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Counting Problems Matchings

Planar Graphs

Planar graphs: can be drawn on plane without crossing edges.



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	General	Bipartite	Planar
Perfect matching	$\stackrel{\scriptstyle{\scriptsize (2)}}{}$		
	[Val87]		
	?		
k-matching			

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Counting Problems Matchings

Perfect Matching-Bipartite Graphs

• Counting: #P-complete [Valiant'87]

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Counting Problems Matchings

Perfect Matching-Bipartite Graphs

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Counting Problems Matchings

Perfect Matching-Bipartite Graphs

- Counting: #P-complete [Valiant'87]
- Approximate counting: in P [Jerrum-Sinclair-Vigoda'04]

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	General	Bipartite	Planar
Perfect matching	8	8	
	[Val87]	[Val87]	
	?	٢	
		[JSV04]	
k-matching			

: approximate, : exact : in P, : #P-complete, ?: Open

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Counting Problems Matchings

Perfect Matching–Planar Graphs

 Counting perfect matching in planar graph is in P!! [Kasteleyn'67]

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	General	Bipartite	Planar
Perfect matching	:	(;)	\odot
	[Val87]	[Val87]	[Kas67]
	?	<u></u>	
		[JSV04]	[Kas67]
k-matching			

: approximate, : exact : in P, : #P-complete, ?: Open

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Counting Problem Matchings

What about non-perfect matching?

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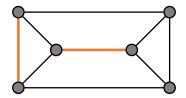
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Counting Problems Matchings

Non-Perfect Matchings: *k*-Matchings

A set of k/2 edges that meets every vertex at most once. Equivalently, a perfect matching on $S \subseteq V$ with |S| = k.



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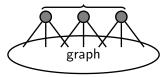
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Counting Problems Matchings

k-Matchings vs. Perfect Matchings

■ Decision/Counting: ∃ reduction from k-matching in G to perfect matching in G'

$$n-2k$$
 dummy nodes



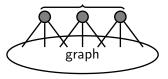
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Counting Problems Matchings

k-Matchings vs. Perfect Matchings

■ Decision/Counting: ∃ reduction from k-matching in G to perfect matching in G'

$$n-2k$$
 dummy nodes



• If G is bipartite, G' is bipartite

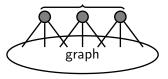
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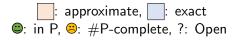


• If G is bipartite, G' is bipartite

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Counting Problem Matchings

	General	Bipartite	Planar
Perfect matching	$\stackrel{\scriptstyle{\scriptsize (2)}}{}$	8	٢
	[Val87]	[Val87]	[Kas67]
	?	٢	٢
		[JSV04]	[Kas67]
k-matching	:	:	
	[Val87]	[Val87]	
	?	٢	
		[JSV04]	



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Counting Problem Matchings

k-Matchings vs. Perfect Matchings

But G' is not planar even if G is planar

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Counting Problem Matchings

k-Matchings vs. Perfect Matchings

- But G' is not planar even if G is planar
- Counting k-matching in planar graph is #P-complete [Jerrum'87]

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Counting Problems Matchings

k-Matchings vs. Perfect Matchings

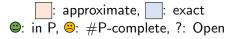
- But G' is not planar even if G is planar
- Counting k-matching in planar graph is #P-complete [Jerrum'87]
- Approximate counting k-matching in planar graph is in P [this work]

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Counting Problem Matchings

	General	Bipartite	Planar
Perfect matching	:		٢
	[Val87]	[Val87]	[Kas67]
	?	٢	٢
		[JSV04]	[Kas67]
k-matching	:	:	æ
	[Val87]	[Val87]	[Jer87]
	?		٢
		[JSV04]	[this]



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Counting Problem Matchings

Main Result: Counting Matching

Theorem

Efficient algorithm to approximately count k-matching (runtime $\approx poly(|V|, k, \log \frac{1}{\epsilon}))$

Planar graphs

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Counting Problems Matchings

Main Result: Counting Matching

Theorem

Efficient algorithm to approximately count k-matching (runtime $\approx poly(|V|, k, \log \frac{1}{\epsilon}))$

- Planar graphs 😄
- Any graph where counting #PM of subgraphs is easy

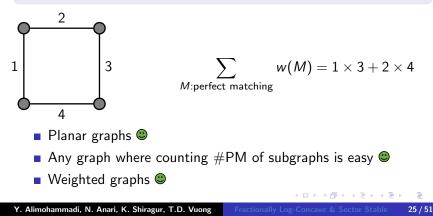
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Counting Problems Matchings

Main Result: Counting Matching

Theorem

Efficient algorithm to approximately count k-matching (runtime $\approx poly(|V|, k, \log \frac{1}{\epsilon}))$



Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Overview

- Background
 - Counting Problems
 - Matchings
- 2 Technique
 - Reduce Counting to Sampling
 - Sampling via Random Walks
 - Fast Mixing From Sector-Stability
- **3** Other Applications

Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Reduce Counting to Sampling

Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Reduce Counting to Sampling

Approximate Counting \equiv Compute \hat{Z} s.t. $\frac{\hat{Z}}{\# k-\text{matchings}} \in [1 - \epsilon, 1]$ Approximate Sampling \equiv Output a *k*-matching according to a distribution that is ϵ -away from the uniform dist. over *k*-matchings

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Reduce Counting To Sampling

Counting \equiv Compute # k-matchings in graph Sampling \equiv Output a random k-matching

■ Approximate Counting ⇔ Approximate Sampling [Jerrum-Vazirani-Vazirani'86]

$$\#\mathsf{M} = \frac{\#\mathsf{M}}{\#\mathsf{M} \text{ contains } 1} \times \frac{\#\mathsf{M} \text{ contains } 1}{\#\mathsf{M} \text{ contains } 1, 2} \cdots$$

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Reduce Counting To Sampling

Counting \equiv Compute # k-matchings in graph Sampling \equiv Output a random k-matching

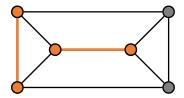
■ Approximate Counting ⇔ Approximate Sampling

$$\#\mathsf{M} = \frac{1}{\mathbb{P}[\mathsf{M} \text{ contains } 1]} \times \frac{1}{\mathbb{P}[\mathsf{M} \text{ contains } 2 \ | \text{ contain } 1]} \cdots$$

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Reduce Counting To Sampling

- Approximate Counting ⇔ Approximate Sampling
- Sample endpoints of k-matchings

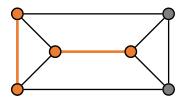


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Reduce Counting To Sampling

- Approximate Counting ⇔ Approximate Sampling
- Sample endpoints of k-matchings
- Given endpoints set S(|S| = k), sample a matching on S

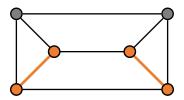


Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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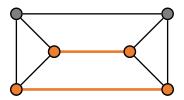


Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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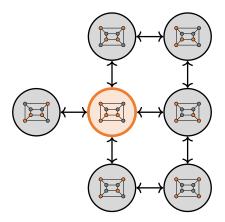


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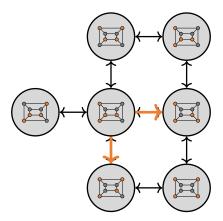
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Sample Endpoints Using Random Walk



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Sample Endpoints Using Random Walk

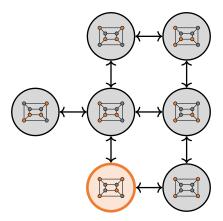


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Sample Endpoints Using Random Walk



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Sample Endpoints Using Random Walk

Start at distribution μ_0 , apply transition rule for T steps to reach desired distribution μ

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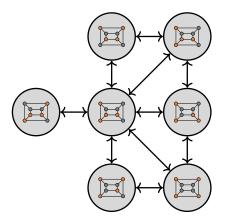
Sample Endpoints Using Random Walk

- Start at distribution μ_0 , apply transition rule for T steps to reach desired distribution μ
- Want: T is small (= poly(|V|, k)) i.e. fast mixing

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Sample Endpoints Using Random Walk

Local Walk: Only allow local moves between S, T that are close i.e. $|S \setminus T| \le 1 \rightarrow$ easy to transition between $S, T \otimes$



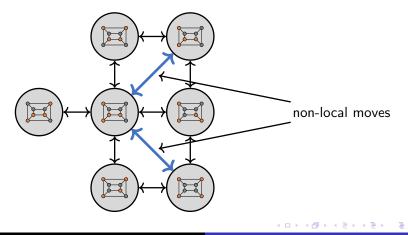
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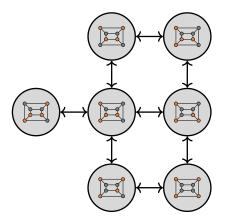


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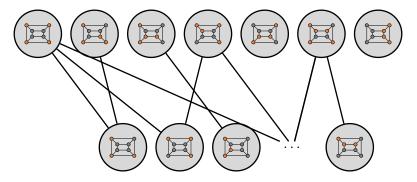
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Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



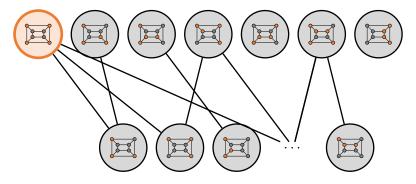
Sample from homogeneous distribution μ over $\binom{[n]}{k}$. **1** Drop an element uniformly at random.

Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



Sample from homogeneous distribution μ over $\binom{[n]}{k}$.

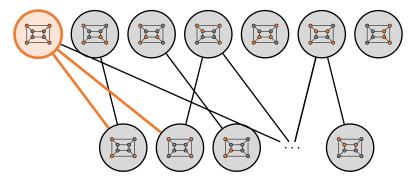
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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



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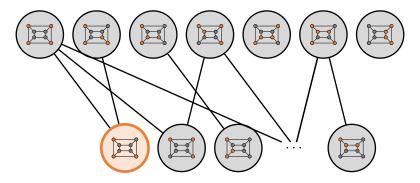
1 Drop an element uniformly at random.

Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



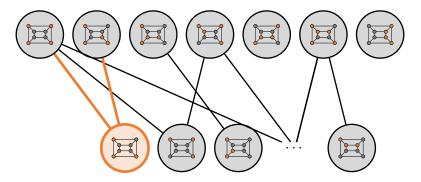
Sample from homogeneous distribution μ over $\binom{[n]}{k}$.

1 Drop an element uniformly at random.

Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



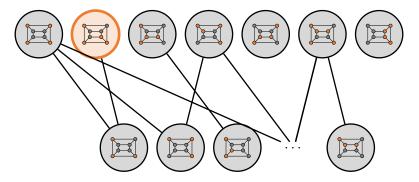
Sample from homogeneous distribution μ over $\binom{[n]}{k}$.

- **1** Drop an element uniformly at random.
- **2** Add an element with probability $\propto \mu$ (resulting set).

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Random Walk $k \leftrightarrow (k-1)$ (1-Step Down-Up Walk)



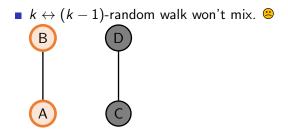
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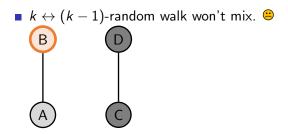
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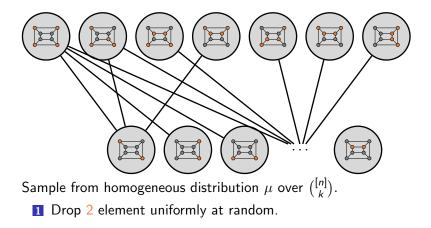
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- $k \leftrightarrow (k-1)$ -random walk won't mix. 🙁
- $k \leftrightarrow (k-2)$ walk does!

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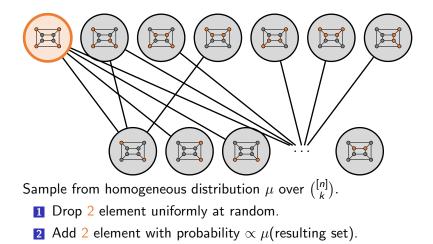
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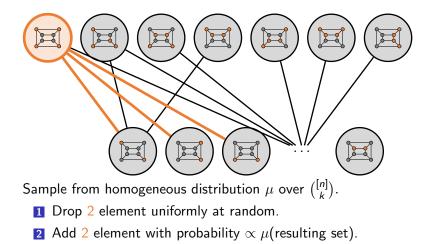
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Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

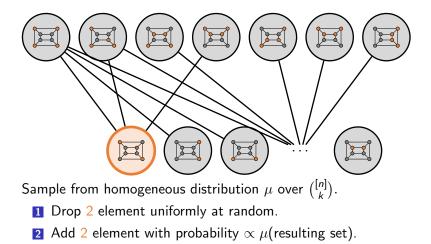
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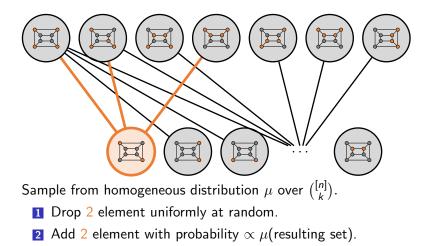
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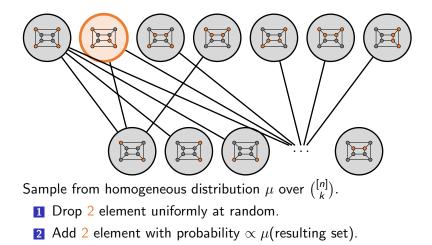
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We can bound mixing time of multi-step down-up walk by proving that the hypergraph associated with μ is a high-dimensional expander.

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High-dimensional expander

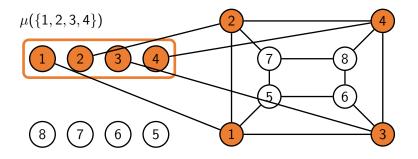
View distribution as weighted hypergraphs

Y. Alimohammadi, N. Anari, K. Shiragur, T.D. Vuong Fractionally Log-Concave & Sector Stable 37/51

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High-dimensional expander

View distribution $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ as weighted *k*-uniform hypergraphs



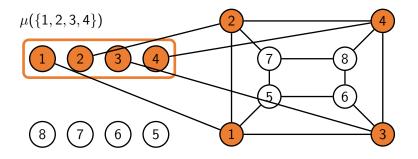
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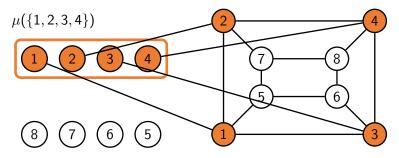
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High-dimensional expander

View distribution $\mu : {\binom{[n]}{k}} \to \mathbb{R}_{\geq 0}$ as weighted *k*-uniform

hypergraphs

 $\label{eq:High-dimensional} \mbox{High-dimensional expansion (HDX): measuring connected-ness of hypergraph$



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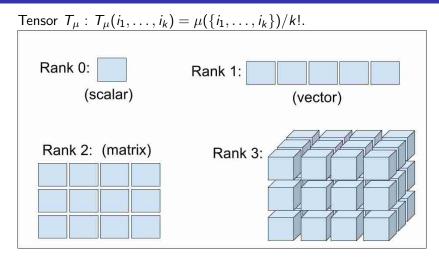
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High-dimensional expander (HDX): tensor view



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High-dimensional expander (HDX): tensor view

Tensor
$$T_{\mu}$$
: $T_{\mu}(i_1, ..., i_k) = \mu(\{i_1, ..., i_k\})/k!$.

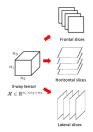
■ k = 2 : (graph) expansion ≡ spectral properties of the adjacency matrix

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High-dimensional expander (HDX): tensor view

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- k > 2 : HDX ≡ spectral properties of all dimension 2 slices & averages of slices of tensor.



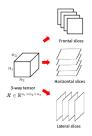
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High-dimensional expander (HDX): tensor view

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- k = 2 : (graph) expansion ≡ spectral properties of the adjacency matrix
- k > 2: HDX \equiv spectral properties of all dimension 2 slices & averages of slices of tensor. slice (link): $\langle T_{\mu}, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{k-2}} \rangle$ average of slice (1-skeleton): $\langle T_{\mu}, (\mathbb{1}/n) \otimes (\mathbb{1}/n) \otimes \cdots \otimes (\mathbb{1}/n) \rangle$



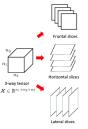
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High-dimensional expander (HDX): tensor view

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Mixing time \equiv spectral properties of transition matrix Mthink M as

"unpacking" of T_{μ} .

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Proof of fast mixing: outline

HDX Spectral Independence

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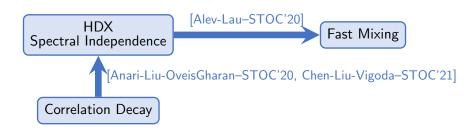
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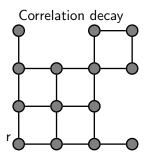
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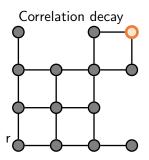
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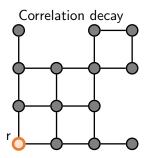
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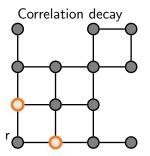


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Proof of fast mixing: outline

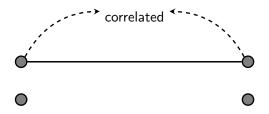


r is only highly correlated with a few "nearby" vertices

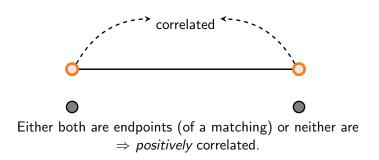
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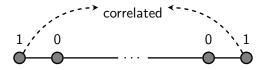
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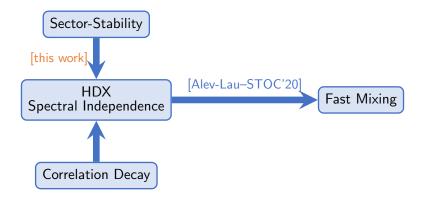
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Reduce Counting to Sampling Sampling via Random Walks Fast Mixing From Sector-Stability

From root-free-ness to fast algorithm: intuition

•
$$\mu \sim {\binom{[n]}{k}} \stackrel{\text{encode}}{\longleftrightarrow} f_{\mu}(z_1, \dots, z_n) = \sum_{S} \mu(S) \prod_{i \in S} z_i$$

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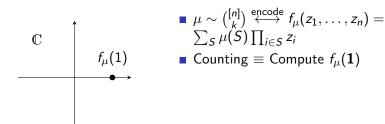
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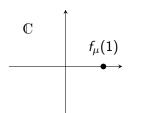
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From root-free-ness to fast algorithm: intuition



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From root-free-ness to fast algorithm: intuition



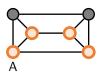
- $\mu \sim {[n] \choose k} \stackrel{\text{encode}}{\longleftrightarrow} f_{\mu}(z_1, \dots, z_n) = \sum_{S} \mu(S) \prod_{i \in S} z_i$
- Counting \equiv Compute $f_{\mu}(\mathbf{1})$
- $f_{\mu}(\mathbf{z}) = 0 \longleftrightarrow \log f_{\mu}(\mathbf{z})$ singular

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From root-free-ness to fast algorithm: intuition



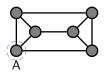
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From root-free-ness to fast algorithm: intuition



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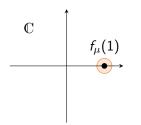
 $\partial_{z_1} f_{\mu}(\mathbf{z}) = \mathbb{P}[A \text{ is endpoint}]$

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From root-free-ness to fast algorithm: intuition



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• No roots "near" $1 \Rightarrow$ "easy" to approximate f_{μ}

 Background
 Reduce Counting to Sampling

 Technique
 Sampling via Random Walks

 Other Applications
 Fast Mixing From Sector-Stabilit

How to make it concrete?

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Generating polynomial

For distribution $\mu : {[n] \choose k} \to \mathbb{R}_{\geq 0}$, let its generating polynomial be

$$f_{\mu}(z_1,\ldots,z_n) = \sum_{S} \mu(S) z^{S} = \sum_{S} \mu(S) \prod_{i \in S} z_i = \langle T_{\mu}, \mathbf{z}^{\otimes k} \rangle$$

where T_{μ} is the tensor defined by $T_{\mu}(i_1, \ldots, i_k) = \mu(\{i_1, \ldots, i_k\})/k!$.

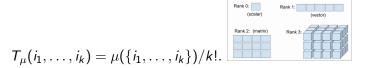
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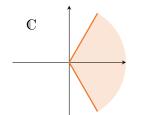
where T_{μ} is the tensor defined by $T_{\mu}(i_1, \ldots, i_k) = \mu(\{i_1, \ldots, i_k\})/k!$.

Question

 f_{μ} no roots near $\mathbb{R}^{n}_{+} \Rightarrow$ Efficient Sampling from μ ?

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Sector stable polynomial



Definition

f is
$$lpha$$
-sector-stable if $f(z)
eq 0$ for z in $S_lpha=\{z\in \mathbb{C}^* \mid |{
m arg}(z)|$

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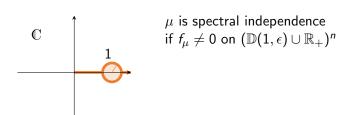
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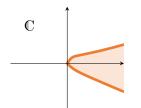
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Generalization



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Generalization



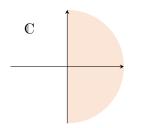
 μ is spectral independence [Chen-Liu-Vigoda–FOCS'21]: if $f_{\mu} \neq 0$ on infinite regions

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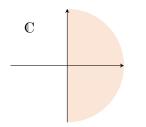
Example of α -Sector-Stable Distributions



 (Non-homogeneous) endpoint distribution is 1-sector-stable (Hurwitz stable) [Hellman-Lieb'72]

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Example of α -Sector-Stable Distributions

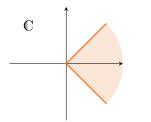


 (Non-homogeneous) endpoint distribution is 1-sector-stable (Hurwitz stable) [Hellman-Lieb'72]

 1-sector stable ≡ half-plane stable (circular region): has been studied by [Lee-Yang'52,Borcea-Branden'09]

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Example of α -Sector-Stable Distributions



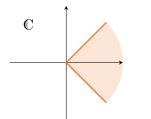
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- Homogenegous endpoint distribution is 1/2-sector-stable [this work]

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Example of α -Sector-Stable Distributions



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- Homogenegous endpoint distribution is 1/2-sector-stable [this work]
- Many things to explore

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Sector stable \Rightarrow Spectral Independence

 Ψ^{cor} : correlation matrix

$$\Psi^{\mathsf{cor}}_{\mu}(i,j) = \mathbb{P}_{\mathcal{S} \sim \mu}[j \in \mathcal{S} \mid i \in \mathcal{S}]$$

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$$\Psi^{\mathsf{cor}}_{\mu}(i,j) = \mathbb{P}_{\mathcal{S} \sim \mu}[j \in \mathcal{S} \mid i \in \mathcal{S}]$$

Theorem (Main technical)

If μ is α -sector stable, then $\forall \lambda \in \mathbb{R}^n_{>0}, \|\Psi^{cor}_{\mu}\|_1 \leq 2/\alpha$.

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Sector stable \Rightarrow Spectral Independence

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If μ is α -sector stable, then $\forall \lambda \in \mathbb{R}^{n}_{\geq 0}$, $\underbrace{\|\Psi^{cor}_{\lambda*\mu}\| \leq 2/\alpha}_{\frac{\alpha}{2} - FLC}$.

Spectral independence $\equiv \{ \| \Psi_{\mu(|S)}^{cor} \|_2 \le O(1) \forall S \}$ If μ is sector stable, then $\lambda * \mu$ (defined by $\lambda * \mu(S) \propto \mu(S) \prod_{i \in S} \lambda_i$) is sector stable Fractional log-concave (FLC) $\equiv \lambda * \mu$ spectrally independent for all external field $\lambda \in \mathbb{R}^n_{\ge 0}$.

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Sector stable \Rightarrow Spectral Independence

Ψ^{cor} : correlation matrix

$$\Psi^{\mathsf{cor}}_{\mu}(i,j) = \mathbb{P}_{\mathcal{S} \sim \mu}[j \in \mathcal{S} \mid i \in \mathcal{S}]$$

Theorem (Main technical)

If μ is α -sector stable, then $\forall \lambda \in \mathbb{R}^{n}_{\geq 0}$,

Spectral independence $\equiv \{ \| \Psi_{\mu(|S)}^{cor} \|_2 \le O(1) \forall S \}$ If μ is sector stable, then $\lambda * \mu$ (defined by $\lambda * \mu(S) \propto \mu(S) \prod_{i \in S} \lambda_i$) is sector stable Fractional log-concave (FLC) $\equiv \lambda * \mu$ spectrally independent for all external field $\lambda \in \mathbb{R}^n_{\ge 0}$.

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Geometry of polynomial view of fractional log-concavity (FLC)

μ is $\alpha\text{-fractionally}\log$ concave \approx

$$f_{\mu}(z_1,\ldots,z_n)^{\frac{1}{k\alpha}} \leq \sum_{i=1}^n p_i z_i^{1/\alpha}$$

with $p_i = \frac{1}{k} \frac{\partial f}{\partial z_i}(\vec{1})$

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Geometry of polynomial view of fractional log-concavity (FLC)

 μ is $\alpha\text{-fractionally}\log$ concave \approx

$$\langle T_{\mu}, \mathbf{z}^{\otimes k} \rangle^{\frac{1}{k\alpha}} = f_{\mu}(z_1, \dots, z_n)^{\frac{1}{k\alpha}} \leq \sum_{i=1}^n p_i z_i^{1/\alpha} \approx \|z\|_{1/\alpha}^{1/\alpha}$$

with $p_i = \frac{1}{k} \frac{\partial f}{\partial z_i}(\vec{1})$

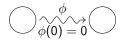
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Proof of main technical theorem

Complex Analysis

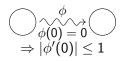


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Proof of main technical theorem

Complex Analysis



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Proof of main technical theorem

Complex Analysis

• Write $\|\Psi_{\mu}^{cor}(i,\cdot)\|_1 = \phi'(0)$ for ϕ' holomorphic

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Proof of main technical theorem

- Complex Analysis
- Write $\|\Psi^{cor}_{\mu}(i,\cdot)\|_1 = \phi'(0)$ for ϕ' holomorphic
- Use Schwarz's lemma to bound $\phi'(0)$.

Overview

- 1 Background
 - Counting Problems
 - Matchings
- 2 Technique
 - Reduce Counting to Sampling
 - Sampling via Random Walks
 - Fast Mixing From Sector-Stability
- 3 Other Applications

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 Local Markov chains to sample from determinantal point processes (DPP).

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 Local Markov chains to sample from determinantal point processes (DPP).

 $\mu(S) = \det(L_{S,S}) \forall S \subseteq [n].$

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Local Markov chains to sample from determinantal point processes (DPP).

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• Local Markov chains to sample from nonsymmetric determinantal point processes (DPP) [Gartrell-Brunel'20] with kernel *L* when $L + L^T$ PSD.

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 Local Markov chains to sample from nonsymmetric determinantal point processes (DPP).

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 Local Markov chains to sample from nonsymmetric determinantal point processes (DPP).

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- Local Markov chains to sample from nonsymmetric determinantal point processes (DPP).
- Efficient algorithm for counting/sampling DPP intersects with partition constraints

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- Local Markov chains to sample from nonsymmetric determinantal point processes (DPP).
- Efficient algorithm for counting/sampling DPP intersects with partition constraints

Generally: the intersection of Rayleigh matroid and partition matroid of constantly many partitions.

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- Local Markov chains to sample from nonsymmetric determinantal point processes (DPP).
- Efficient algorithm for counting/sampling DPP intersects with partition constraints
- Fast mixing \Rightarrow MAP-inference via local search [Anari-V.'21]

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