On the Complexity of Sampling Redistricting Plans

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Based on joint work with Moses Charikar, Paul Liu and Tianyu Liu

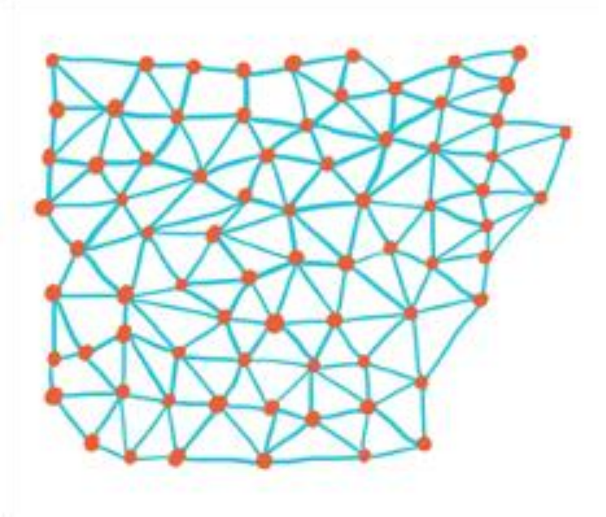
Outline

1. Redistricting problem
2. Sampling redistricting plans using Markov chains
   - Spanning tree distribution
   - Recombination (ReCom) chain
3. Slow mixing of ReCom on natural subgraph of the grid
4. Fixing ReCom
   - Fast mixing of relaxed ReCom
   - Enforcing balanced constraint

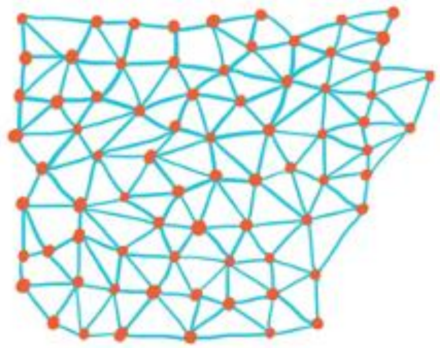
Redistricting

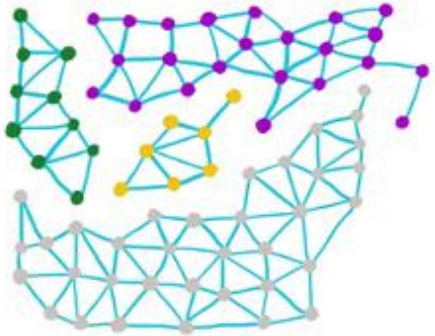
Partitioning map into districts = partitioning vertices of dual graph

Property of graph: planar, low degree (think: grid/subgraph of grid)



Redistricting





Partitioning map into districts = partitioning vertices of dual graph

Desirable properties for partitioning $P_1, \ldots, P_k \subseteq V$

- Balanced: equal population $|P_i| \approx |V|/k$
- Contiguous: $G[P_i]$ is connected
- Compact: few edges between different parts (small cut)

Gerrymandering

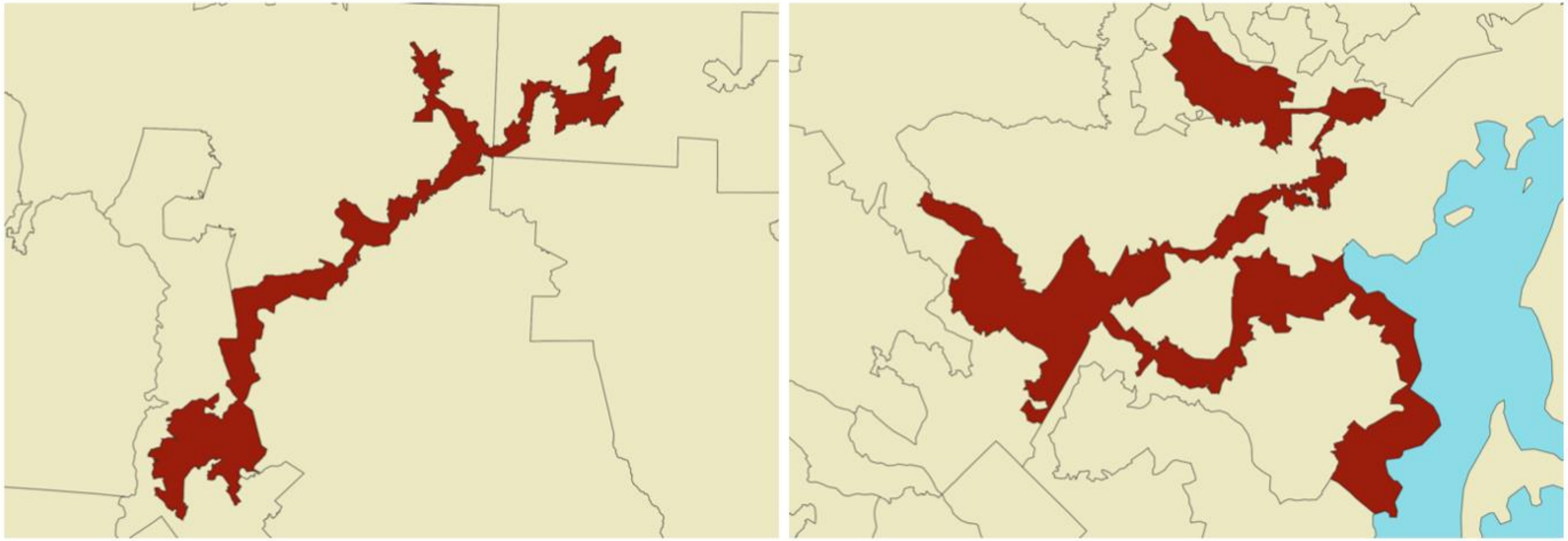


Figure 4: Examples of scapegoating: North Carolina's 12th district and Maryland's 3rd, circa 2013.

Detecting gerrymandering

Is a given redistricting plan $P = (P_1, \ldots, P_k)$ an instance of gerrymandering?

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Solution: sample many redistricting plans from some bipartisan agreed-upon distribution, then show that P is "similar" to these plans

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Which distribution to sample from?

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Idea 2: few cut edges = many spanning tree in each part

[Procaccia—Tucker-Foltz—SODA'21]

Spanning tree distribution

Let $T(P_i) = \#\,\{\text{spanning trees in } G[P_i]\}$.

Distribution μ over k-partition of graph G s.t.

$$\mu(P_1, \ldots, P_k) \;\propto\; \prod T(P_i)$$

$\mu^{balanced}$: μ restricted to balanced partititoning

ReCom chain [Deford-Duchin-Solomon'19]

Current partition $(P_1, \ldots, P_k)$

- Merge step: merge two parts P_i, P_j
- Split step: split the union to P'_i, P'_j

Can choose the probability of merge and split so that the Markov chain converges to $\pi \equiv \mu^{balanced}$

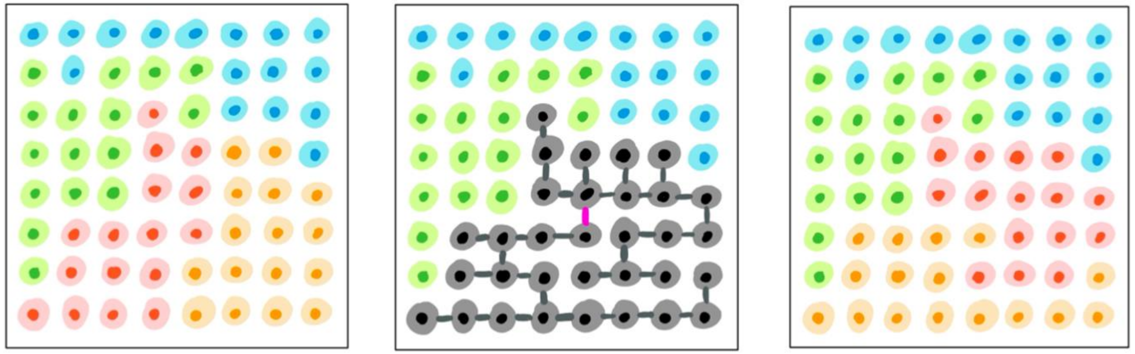

Figure 11: The basic recombination step: two districts are merged, a spanning tree is chosen, a balance edge is selected and cut, leaving two new districts.

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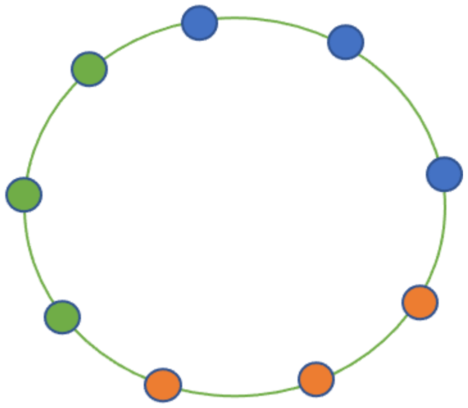
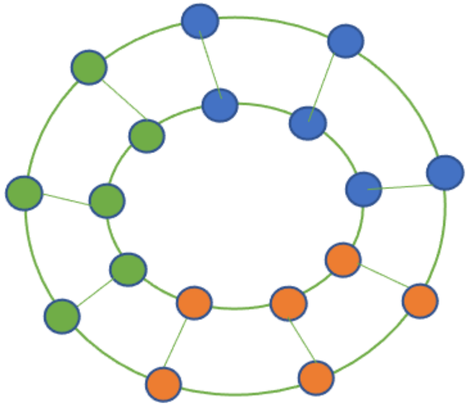
   - c-weighted spanning tree distribution
   - Proportion of balanced partitionings

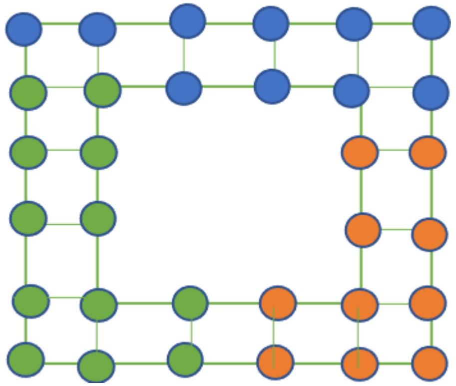
Slow mixing of ReCom



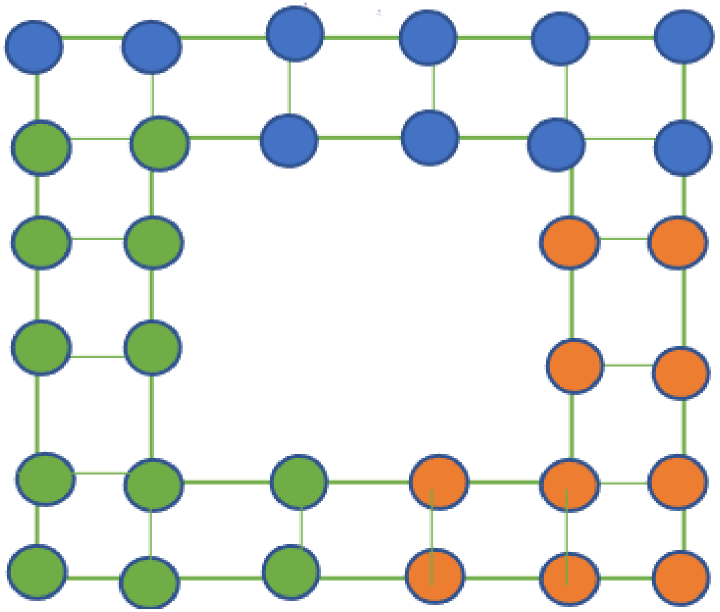
ReCom is not connected on the single cycle



ReCom is connected on the double cycle, but takes $\exp(\Omega(n))$ to mix



ReCom is connected on grid-with-a-hole, but takes $\exp(\Omega(n))$ to mix



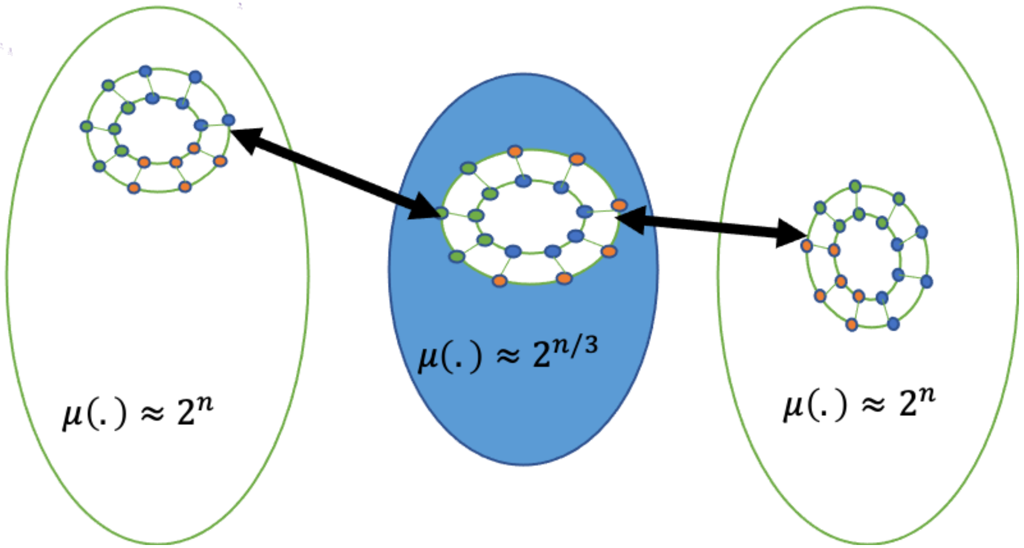
Natural example of grid-with-a-hole: map around a lake

Double cycle: proof intuition

Show existence of small cutset:

A set C of states (partitionings) whose removal disconnect two states A and B s.t.

$$\frac{\pi(A)}{\pi(C)} \approx \frac{\pi(B)}{\pi(c)} = \exp(\Omega(n))$$



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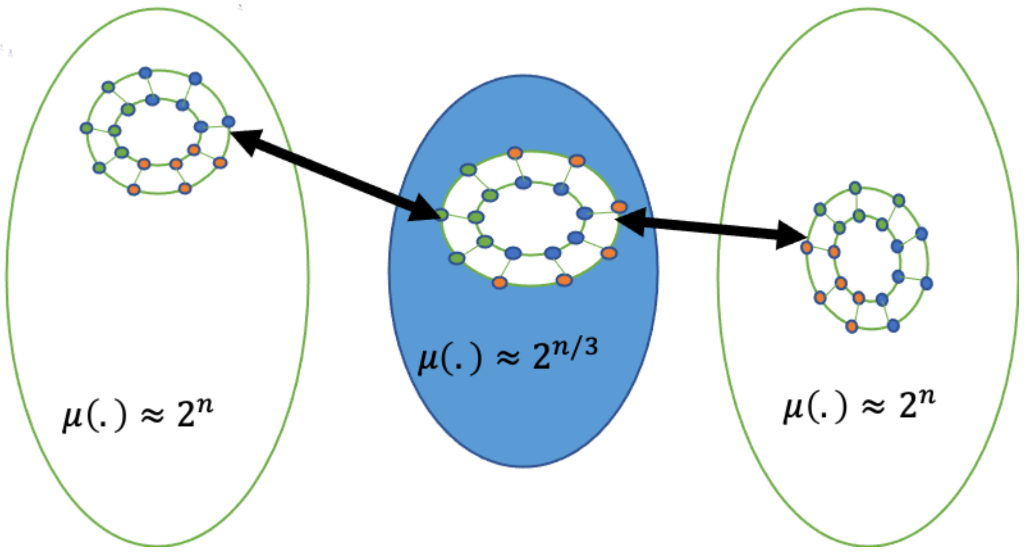
Implies conductance $< \exp(-\Omega(n))$

thus slow mixing



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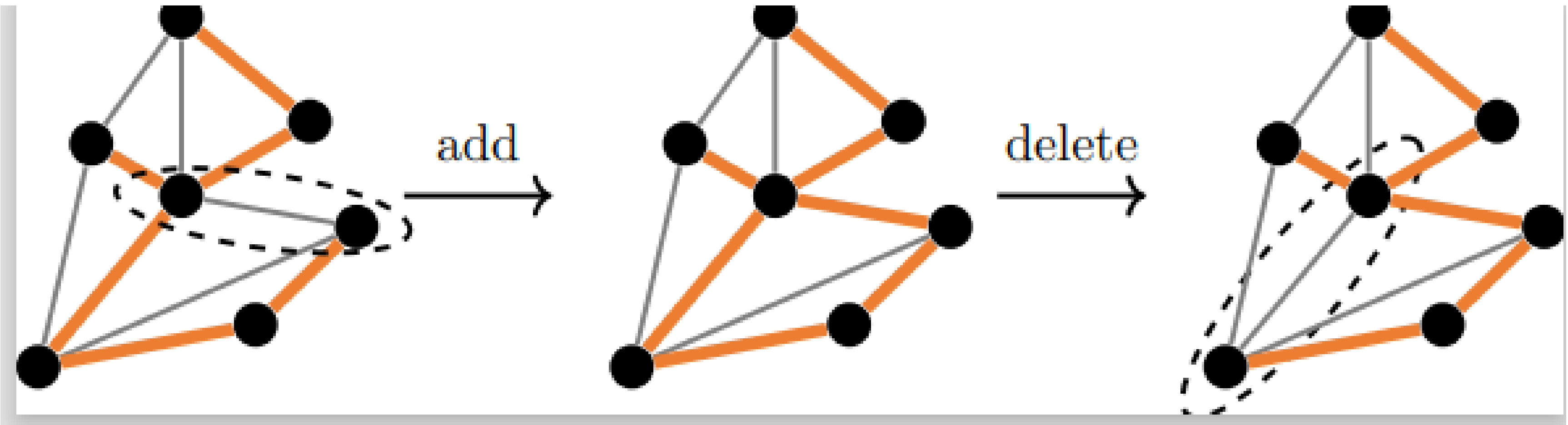
“Relaxed”-ReCom on forests

Current state $(F_1, \dots, F_k)$ where F_i is a tree in $G[P_i], P_i = V(F_i)$

- Merge step: merge two trees F_i, F_j by adding an edge
- Split step: split into two new trees F'_i, F'_j by removing an edge

This is precisely “Up-down walk on forest” [Anari-Liu-OveisGharan-Vinzant-Vuong--STOC’21] and runs in $O(|E| \log |E|)$ time

"Relaxed"-ReCom on forests



Enforcing balanced constraint

Q: How to enforce balanced constraint?

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On mxn grid-graph, experimental evidence that for $k = O(1)$

$$\Pr_{\mu}[balanced] = 1/poly(m, n)$$

Proof for the case m $= O(1)$, for double cycle/grid-with-a-hole.

Soft balanced constraint

C-weighted spanning tree distribution: μ^c over k-partition of graph G s.t.

$$\mu^c(P_1, \ldots, P_k) \propto \prod T(P_i)\, |P_i|^c$$

$\mu^c = \mu$ if c=0 $\mu^c = \mu$ if c=0

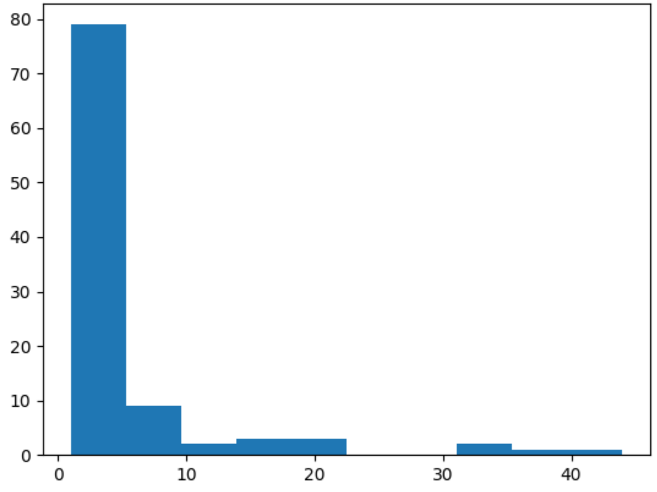
$\mu^c = \mu^{balanced}$ if $c = \infty$

Soft balanced constraint

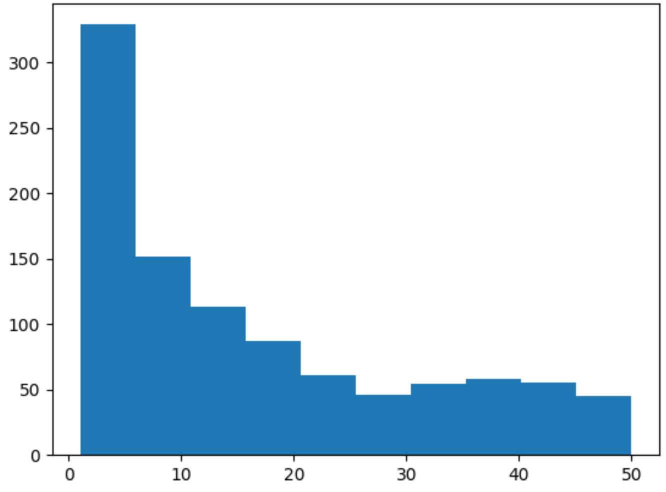
Similar up-down walk can sample from μ^c in time $\tilde{O}(|V|^{O(c)}|E|)$

(Analysis using comparison with μ^0)

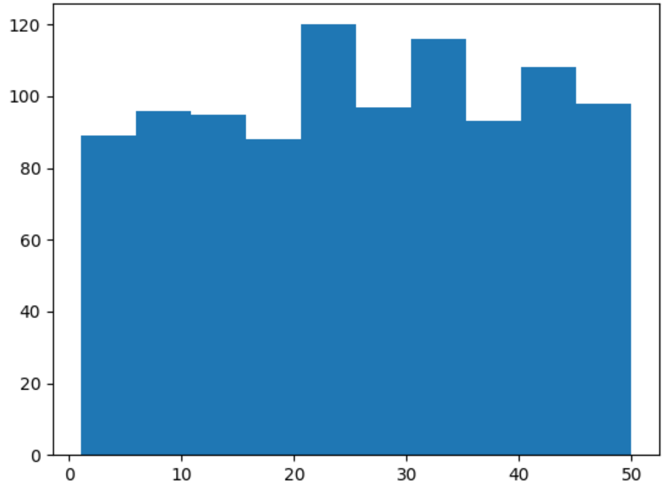
Proportion of balanced constraint



C=0



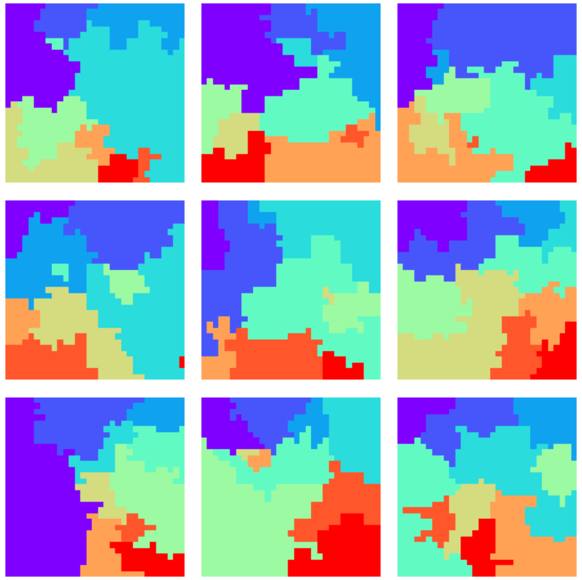
C=1



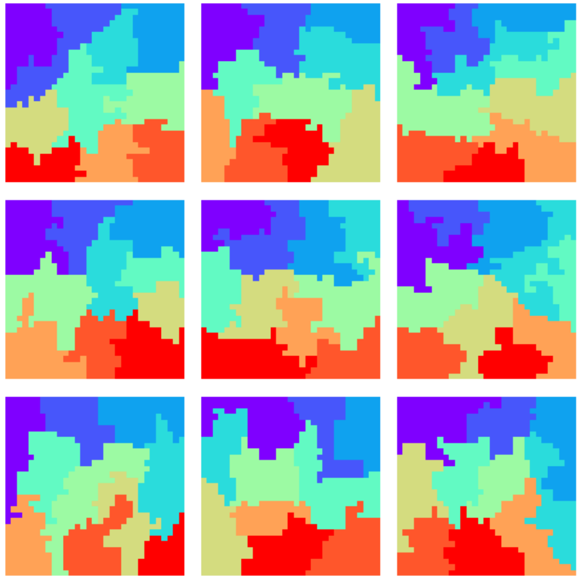
C=2

10x10 grid graph, k=2, histogram for size of minimum connected component

Proportion of balanced constraint



C=2



C=20

30x30 grid graph, k=10, example of sampled partitionings for different values of C