An Extension of Plücker Relations with Applications to Subdeterminant Maximization

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Problem Definition

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- Output: Compute

$$\text{maxdet}_k(A) := \max\left\{ |\det(A_{I,J})| \;\middle|\; I \in \binom{[m]}{k}, J \in \binom{[n]}{k} \right\}$$

Main result

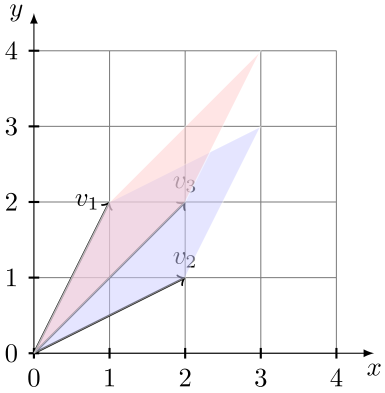
Theorem (AV**'20)**

There is a polynomial time algorithm that on input $A \in \mathbb{R}^{m \times n}$, outputs sets of indices $I \in \binom{[m]}{k}$ and $J \in \binom{[n]}{k}$ guaranteeing

$$k^{O(k)} \cdot |\det(A_{I,J})| \geq \text{maxdet}_k(A).$$

Background

- Special case: $k = \min\{m, n\}$ i.e. *maximal* subdeterminant. Equivalent formulation: largest volume simplex problem.



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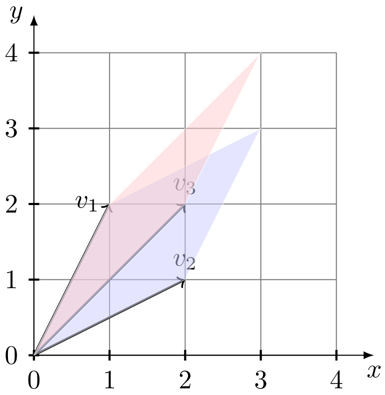

- [Nik'15]: $2^{O(k)}$-approximation
  [Di+14]: matching lower bound.

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[LSV'86,Mat'13]

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[NT'14]: $\log^{3/2} n$-approx of herdisc using γ_2.

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General case

Consider

$$A := \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

exchange only 1 row/column, stuck with $\det = 0$

$\Rightarrow$ need 2 exchanges per iteration!

General case: Local Search

Parameter $\alpha < 1$.

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Runtime: $\log_{1/\alpha}(OPT/|\det(A_{S^0})|) \Rightarrow$ start with $\mathrm{poly}\{n, m\}^k$-approximation

Local Search: (r, α) local-maxima

- For indices $S = (S_{\text{row}}, S_{\text{col}})$, $T = (T_{\text{row}}, T_{\text{col}})$, let
$$d(S, T) := |S \Delta T|/2 = |S_{\text{row}} \Delta T_{\text{row}}|/2 + |S_{\text{col}} \Delta T_{\text{col}}|/2$$
- Let r-neighborhood of S be
$$\mathcal{N}_r(S) := \{T \mid d(S, T) \leq r\}.$$
- S is (r, α)-local maxima iff
$|\det(A_S)| \geq \alpha |\det(A_T)| \forall T \in \mathcal{N}_r(S)$.
Observation: Local Search outputs $(2, \alpha)$-local maxima.

Approximation Ratio: $k^{O(k)}$

Lemma

A $(2, \alpha)$-local maximum S is an $(k/\alpha)^{O(k)}$-approximate global optimum:

$$(k/\alpha)^{O(k)} \cdot |\det(A_S)| \geq \text{maxdet}_k(A).$$

Approximate 2-exchange

Theorem (Exchange Property)

Let S, T be indices of two $k \times k$ submatrices, and assume that $S \neq T$. Then

$$|\det(A_S)| \cdot |\det(A_T)| \leq O(k^2)|\det(A_{S\Delta U})| \cdot |\det(A_{T\Delta U})|$$

for some $U \in \mathcal{E}(S, T)$.

$\mathcal{E}(S, T)$ be set of $U = (U_{\mathrm{row}}, U_{\mathrm{col}})$ satisfying $S\Delta U$, $T\Delta U$ are valid location pairs, and $|U_{\mathrm{row}}| + |U_{\mathrm{col}}| \leq 4$.

Example

$$U_{\text{row}} \left\{ \begin{pmatrix} 8 & 8 & 1 & 6 & 1 & 0 \\ 3 & 8 & 5 & 7 & 2 & 4 \\ 4 & 8 & 9 & 5 & 3 & 3 \\ 4 & 8 & 9 & 2 & 2 & 3 \\ 3 & 7 & 9 & 5 & 3 & 3 \\ 4 & 8 & 6 & 1 & 3 & 0 \end{pmatrix} \right. \to \begin{pmatrix} 8 & 8 & 1 & 6 & 1 & 0 \\ 3 & 8 & 5 & 7 & 2 & 4 \\ 4 & 8 & 9 & 5 & 3 & 3 \\ 4 & 8 & 9 & 2 & 2 & 3 \\ 3 & 7 & 9 & 5 & 3 & 3 \\ 4 & 8 & 6 & 1 & 3 & 0 \end{pmatrix}$$

Figure: $U \in \mathcal{E}(S, T), U_{\text{col}} = \emptyset$

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- Substitute $S :=$ $(2, \alpha)$-local maxima. T arbitrary.
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$$|\det(A_T)| \leq O(k^2) \frac{1}{\alpha} |\det(A_{T\Delta U})|$$

- Observe: $d(T\Delta U, S) \leq d(T, S) - 1$

(Continue)

- Substitute $T = OPT$. Note: $d(T, S) \leq 2k$.

$$|\det(A_T)| \leq O(\frac{k^2}{\alpha})|\det(A_{T_1})| \leq (O(\frac{k^2}{\alpha}))^2|\det(A_{T_2})|$$
$$\leq \cdots \leq (O(\frac{k^2}{\alpha}))^{2k}|\det(A_{T_{2k}})|,$$

where $T_i = T_{i-1}\Delta U_i$. Note

$$d(T_{2k}, S) \leq d(T_{2k-1}, S) - 1 \leq \cdots \leq 2k - 2k = 0$$

so $T_{2k} = S$.

Proof of Exchange Property

- *Classical* Plucker relation:

$$\det(A_{[m],S})\det(A_{[m],T}) = \sum_{i \in S \setminus T} \delta_j^i \det(A_{[m],S\Delta\{i,j\}}) \cdot \det(A_{[m],T\Delta\{i,j\}})$$

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- *2-dimensional* Plucker relation (new!):
  Similar algebraic expression, but involves terms of form

$$\det(A_{S_{\mathrm{row}}\Delta U_{\mathrm{row}}, S_{\mathrm{col}}\Delta U_{\mathrm{col}}}) \cdot \det(A_{T_{\mathrm{row}}\Delta U_{\mathrm{row}}, T_{\mathrm{col}}\Delta U_{\mathrm{col}}})$$

  where $U = (U_{\mathrm{row}}, U_{\mathrm{col}}) \in \mathcal{E}(S, T)$.

Plucker relation

- Classical:

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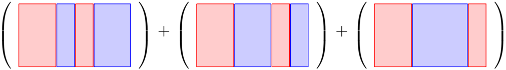

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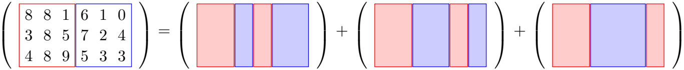
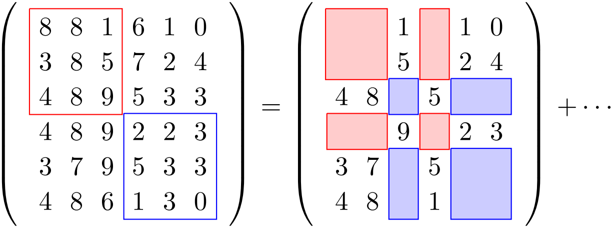


- 2-dimensional:



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2-dimensional:

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Open problems

- 2-dim Plucker relation $\longleftrightarrow$ relations between size-k Pfaffians of
$$X = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}$$
General skew symmetric Y:
no such relation, but Exchange Property probably exist
- $2^{O(k)}$-approximation?
$\Rightarrow O(1)$-approxmiation for $\text{detlb}(A)$.